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A note on extensions of dynamical systems from uniform spaces

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Abstract

This paper is devoted to study a special kind of extensions of dynamical systems which are introduced by considering completions of totally bounded uniform spaces. It is proved that transitivity, minimality and chaos (in Devaney's sense) are carried over these class of extensions.

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1. Introduction

Let X be a compact Hausdorff space. The triple (X, φ, Σ), where φ is a continuous function from X into itself and Σ is the semigroup $\mathbb{N}$ of the non-negative integers or the group $\mathbb{Z}$ of the integer if φ is an homeomorphism, is called a (*discrete*) *dynamical system*. As usual, X is said to be the phase space.

Given two dynamical systems (X, φ, Σ) and $(K, \tilde{\varphi}, \Sigma)$, if there is a continuous surjection (respectively, a homeomorphism)

$$p : K \to X,$$

such that the following diagram

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$$\begin{array}{ccc} K & \xrightarrow{\tilde{\varphi}} & K \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\varphi} & X \end{array}$$

commutes, we say that the dynamical system $(K, \tilde{\varphi}, \Sigma)$ is an extension of (X, φ, Σ) or, equivalently, that (X, φ, Σ) is a factor of $(K, \tilde{\varphi}, \Sigma)$ (respectively, (X, φ, Σ) and $(K, \tilde{\varphi}, \Sigma)$ are topologically conjugates). In general, the dynamical properties of a dynamical system are preserved under factors, but, usually, it is not an easy problem to determine when, given an extension $(K, \tilde{\varphi}, \Sigma)$ of a dynamical system (X, φ, Σ) and a dynamical property $\mathcal{P}$ of (X, φ, Σ), the extension $(K, \tilde{\varphi}, \Sigma)$ also enjoys $\mathcal{P}$. Most of the discussions in the literature on extensions of dynamical system concern extensions arising from a construction like the skew-product [10].

Recently Pennings and Peters introduced an interesting new kind of extensions by using the spectrum of a complex C^*-algebra. To be precise, if $\mathcal{A}$ is a Σ-invariant C^*-algebra of bounded complex-valued functions containing the ring $C(X)$ of all complex-valued continuous functions, then $(\widehat{X}, \hat{\varphi}, \Sigma)$, where $\widehat{X}$ is the spectrum of $\mathcal{A}$ and $\hat{\varphi}$ is the map given by

$$\hat{\varphi}(\hat{x})(f) = \hat{x}(f \circ \varphi), \quad \hat{x} \in \widehat{X},$$

is an extension of (X, φ, Σ) [11] (here Σ-invariant means that $f \circ \varphi^k \in \mathcal{A}$ whenever $f \in \mathcal{A}$ and $k \in \Sigma$). These kind of extensions have been studied later in [12,6,7,9].

In general, topological transitivity and minimality of (X, φ, Σ) are not inherited by $(K, \hat{\varphi}, \Sigma)$. Specifically, if $\mathcal{A}$ is the C^*-algebra generated by the bounded functions which are continuous off a finite set, and each fiber of φ is finite, then (X, φ, Σ) can be minimal without $(\widehat{X}, \hat{\varphi}, \Sigma)$ being topologically transitive; for instance, each characteristic function of a singleton in X represents an obstruction to the topological transitivity of $(\widehat{X}, \hat{\varphi}, \Sigma)$ (see [11, Example II.8]).

The main result in [11] states that if $\mathcal{A}$ is a C^*-algebra containing $C(X)$ and is topologically generated by a subset of functions with at most a finite number of points of discontinuity and contains no characteristic functions of singletons, then for every $x \in X$ and every $k \in p^{-1}(x)$, the orbit of k is dense whenever the orbit of x is dense. As a direct corollary, it follows that topological transitivity (respectively, minimality) passes from (X, φ, Σ) to $(\widehat{X}, \hat{\varphi}, \Sigma)$. In this paper we consider a wide class of C^*-algebras, namely C^*-algebras $\mathcal{A}$ which are topologically generated by a subset of functions each of which is continuous off a dispersed set and has no characteristic functions of singletons, and we prove that minimality, transitivity and chaos (in Devaney's sense) are carried over the extension $(\widehat{X}, \hat{\varphi}, \Sigma)$. We recall to the reader that a set A is dispersed if every subset B of A contains an isolated point. Our approach is methodologically different from the one of Pennings and Peters and it is based in [9]: the phase space of the extension can be obtained as the completion of a totally bounded uniform space.

2. Preliminaries and notation

Let $(Y, \mathcal{V})$ be a uniform space. If the uniformity $\mathcal{V}$ is the weak uniformity induced by a family $\mathcal{B}$ of complex-valued functions, we shall denote by $\tau(\mathcal{V})$ the topology (on Y) induced by $\mathcal{V}$ and, depending on the context, either by $Y_{\mathcal{V}}$ or by $Y_{\mathcal{B}}$ the topological space $(Y, \tau(\mathcal{V}))$ or the uniform space $(Y, \mathcal{V})$. Y denotes the underlying set of $(Y, \mathcal{V})$.

Let (X, φ, Σ) be a dynamical system, where X is a compact topological space. Let $\mathcal{L}$ be the only admissible uniform structure on X [8, 15H]. Let $\mathcal{A}$ be a complex Σ-invariant C^*-algebra with $C(X) \subset \mathcal{A} \subset B(X)$, where $B(X)$ is the ring of all bounded complex-valued functions on X. Let $\mathcal{U}_{\mathcal{A}}$ be the uniformity generated by $\mathcal{A}$, that is, the coarsest uniformity that make all functions in $\mathcal{A}$ uniformly continuous. Then $\mathcal{L} \subset \mathcal{U}_{\mathcal{A}}$ and being all functions in $\mathcal{A}$ bounded, the uniformity $\mathcal{U}_{\mathcal{A}}$ is totally bounded. Let $(K, \widetilde{\mathcal{U}}_{\mathcal{A}})$ be the uniform completion of $(X, \mathcal{U}_{\mathcal{A}})$. Notice that, since $\mathcal{U}_{\mathcal{A}}$ is a totally bounded uniformity, $(K, \widetilde{\mathcal{U}}_{\mathcal{A}})$ is a compact uniform space [5, Corollary 8.3.17]. Remark that $\tau(\mathcal{U}_{\mathcal{A}})$ is the topology induced on X by $\tau(\widetilde{\mathcal{U}}_{\mathcal{A}})$.

Since $\mathcal{U}_{\mathcal{A}}$ is finer than $\mathcal{L}$ and $X_{\mathcal{L}}$ is compact, we obtain that the identity mapping i from $X_{\mathcal{A}}$ onto $X_{\mathcal{L}}$ admits a continuous extension $i_{\mathcal{A}}$ from $K_{\mathcal{A}}$ into $X_{\mathcal{L}}$ [5, Theorem 8.3.10]. Using this result, it is possible to prove that there exists a continuous function $\varphi_{\mathcal{A}}$ such that the following diagram

$$\begin{tikzcd} K_{\mathcal{A}} \arrow[r, "\varphi_{\mathcal{A}}"] \arrow[d, "i_{\mathcal{A}}"'] & K_{\mathcal{A}} \arrow[d, "i_{\mathcal{A}}"] \\ X_{\mathcal{L}} \arrow[r, "\varphi"] & X_{\mathcal{L}} \end{tikzcd}$$

commutes, that is, $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ is an extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$ [9]. For emphasizing the role of the algebra $\mathcal{A}$, the extension $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ is called *an $\mathcal{A}$-extension*. By the definition of $K_{\mathcal{A}}$, the Banach space of complex-valued functions on $K_{\mathcal{A}}$ can be identified with $\mathcal{A}$. In addition, the Gelfand–Naĭmark's Theorem says us that $C(K_{\mathcal{A}})$ coincides also with $\mathcal{A}$. So, by the Banach–Stone's Theorem, there exists an homeomorphism ϕ from $\widehat{X}$ onto $K_{\mathcal{A}}$ such that the restriction of ϕ to X is the identity. Bearing this fact in mind and using the description of $\widehat{X}$ and $\hat{\varphi}$ given in [11, Proposition II.2 and Remark II.5], it is straightforward to prove the following theorem.

Theorem 2.1. $(\widehat{X}, \hat{\varphi}, \Sigma)$ *and* $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ *are topologically conjugates.*

The previous theorem is the key point of our approach: the phase space of the extension obtained by Pennings and Peters is the completion of a totally bounded uniform space. This fact permits us to apply uniform and topological techniques instead of methods from Functional Analysis in the context of extensions defined from $C^\star$-algebras.

Our terminology is standard. For notation not defined here the reader can see [5,14]. From now on, to avoid trivialities, we suppose that X has no isolated points.

3. Topologically transitive and minimal extensions

As usual, given a dynamical system (X, φ, Σ) and $x \in X$, the orbit of x is the set $\{\varphi^n(x)\colon n \in \Sigma\}$ and it is denoted by $\mathcal{O}_\varphi(x)$. A point $x \in X$ is periodic if there exists $n \in \mathbb{N}$ such that $\mathcal{O}_\varphi(x) = \{x, f(x), f^2(x), \dots, f^n(x)\}$. A dynamical system (X, φ, Σ) is said to be *topologically transitive* if for all nonempty open subsets U and V of X there exists a nonnegative integer n such that $\varphi^n(U) \cap V \neq \emptyset$, and it is said to be *minimal* if $\mathcal{O}_\varphi(x)$ is dense in X, for every $x \in X$. If X is separable and second category metric space with no isolated points then (X, φ, Σ) is transitive if and only if there exists a point $x \in X$ such that $\mathcal{O}_\varphi(x)$ is dense in X (see [13]).

Given a dynamical system $(X_\mathcal{L}, \varphi, \Sigma)$ we introduce a new kind of $\mathcal{A}$-extension named $\mathcal{A}_D$-extension. Let $\mathcal{A}$ be a complex Σ-invariant C^*-algebra such that $C(X_\mathcal{L}) \subset \mathcal{A} \subset B(X)$. We say that the $\mathcal{A}$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$ induced by $\mathcal{A}$ is an $\mathcal{A}_D$-extension if $\mathcal{A}$ is topologically generated by a family $F \subset B(X)$ of functions such that the set

$$D_f = \{x \in X \mid f \text{ is discontinuous at } x\}$$

is dispersed in $X_\mathcal{L}$, for every $f \in F$.

We shall denote by $D(X_\mathcal{L})$ the subalgebra of $B(X)$ of all functions f in $B(X)$ such that the set D_f is dispersed in $X_\mathcal{L}$.

Notice that the family Φ of all extensions of a dynamical system $(X_\mathcal{L}, \varphi, \Sigma)$ admits a partial order: given two elements $(K_{\mathcal{A}_1}, \varphi_{\mathcal{A}_1}, \Sigma)$ and $(K_{\mathcal{A}_2}, \varphi_{\mathcal{A}_2}, \Sigma)$ in Φ, we will write $(K_{\mathcal{A}_1}, \varphi_{\mathcal{A}_1}, \Sigma) \leqslant_\Phi (K_{\mathcal{A}_2}, \varphi_{\mathcal{A}_2}, \Sigma)$ if $(K_{\mathcal{A}_2}, \varphi_{\mathcal{A}_2}, \Sigma)$ is an extension of $(K_{\mathcal{A}_1}, \varphi_{\mathcal{A}_1}, \Sigma)$. The following theorem permits us to prove that the order $\leqslant_\Phi$ induces a lattice structure in the family of all $\mathcal{A}_D$-extensions of a dynamical system.

Theorem 3.1. *Let $(X_\mathcal{L}, \varphi, \Sigma)$ be a dynamical system. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be two C^*-algebras such that $C(X_\mathcal{L}) \subset \mathcal{A}_i \subset D(X_\mathcal{L})$ and $(K_{\mathcal{A}_i}, \varphi_{\mathcal{A}_i}, \Sigma)$ is an $\mathcal{A}_{i_D}$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$, for each $i = 1, 2$. Then, $(K_{\mathcal{A}_2}, \varphi_{\mathcal{A}_2}, \Sigma)$ is an extension of $(K_{\mathcal{A}_1}, \varphi_{\mathcal{A}_1}, \Sigma)$ if and only if $\mathcal{A}_1 \subset \mathcal{A}_2$.*

Proof. *Sufficiency.* Let $(K_{\mathcal{A}_{i_D}}, \varphi_{\mathcal{A}_{i_D}}, \Sigma)$ be two $\mathcal{A}_D$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$, for $i = 1, 2$, with $\mathcal{A}_1 \subset \mathcal{A}_2$. By [5, Theorem 8.3.10], the inclusion $\mathcal{A}_1 \subset \mathcal{A}_2$, implies that the identity map

$$\bar{\imath} : X_{\mathcal{A}_2} \to X_{\mathcal{A}_1} \subset K_{\mathcal{A}_1}$$

admits a uniformly continuous extension, say $\tilde{\imath}$, to $K_{\mathcal{A}_2}$ such that $\tilde{\imath}|_X = \bar{\imath}$. Because

$$\tilde{\imath}(X) \subset \tilde{\imath}(K_{\mathcal{A}_2}) \subset K_{\mathcal{A}_1}$$

and $\tilde{\imath}(K_{\mathcal{A}_2})$ is a closed subset of $K_{\mathcal{A}_1}$, we obtain

$$K_{\mathcal{A}_1} = \mathrm{cl}_{K_{\mathcal{A}_1}} X \subset \mathrm{cl}_{K_{\mathcal{A}_1}} \tilde{\imath}(K_{\mathcal{A}_2}) = \tilde{\imath}(K_{\mathcal{A}_2}) \subset K_{\mathcal{A}_1}.$$

Therefore $\tilde{\imath}$ is a continuous surjection. In addition, since both $\varphi_{\mathcal{A}_2}|_X$ and $\varphi_{\mathcal{A}_1}|_X$ coincides with φ we have that the diagram

$$\begin{tikzcd} K_{\mathcal{A}_2} \arrow[r, "\varphi_{\mathcal{A}_2}"] \arrow[d, "\tilde{\imath}"'] & K_{\mathcal{A}_2} \arrow[d, "\tilde{\imath}"] \\ K_{\mathcal{A}_1} \arrow[r, "\varphi_{\mathcal{A}_1}"] & K_{\mathcal{A}_1} \end{tikzcd}$$

commutes. Thus $(K_{\mathcal{A}_2}, \varphi_{\mathcal{A}_2}, \Sigma)$ is an extension of $(K_{\mathcal{A}_1}, \varphi_{\mathcal{A}_1}, \Sigma)$.

Necessity. Let $\tilde{\imath}$ be as above and suppose that the diagram

$$\begin{tikzcd} K_{\mathcal{A}_2} \arrow[r, "\varphi_{\mathcal{A}_2}"] \arrow[d, "\tilde{\imath}"'] & K_{\mathcal{A}_2} \arrow[d, "\tilde{\imath}"] \\ K_{\mathcal{A}_1} \arrow[r, "\varphi_{\mathcal{A}_1}"] \arrow[d, "i_{\mathcal{A}_1}"'] & K_{\mathcal{A}_1} \arrow[d, "i_{\mathcal{A}_1}"] \\ X_{\mathcal{L}} \arrow[r, "\varphi"] & X_{\mathcal{L}} \end{tikzcd}$$

commutes. Let $f \in \mathcal{A}_1$. By [8, 6H] it is sufficient to prove that, given a point $k \in K_{\mathcal{A}_2} \setminus X$, there exists a function g defined on $X \cup \{k\}$ such that $g|_X = f$ and $(f(x_\delta))_{\delta \in D}$ converges to $g(k)$ for every net $(x_\delta)_{\delta \in D} \subset X$ converging to k in $K_{\mathcal{A}_2}$. To see this, let $m \in K_{\mathcal{A}_1}$ be such that $k \in \tilde{\imath}^{-1}(m)$. Since $(x_\delta)_{\delta \in D}$ converges to $\tilde{\imath}(k)$ in $K_{\mathcal{A}_1}$, the net $(f(x_\delta))_{\delta \in D}$ converges to $f(\tilde{\imath}(k))$. So, the function defined as

$$g(t) = \begin{cases} f(t) & \text{if } t \in X, \\ f(k) & \text{if } t = k \end{cases}$$

enjoys the properties what we claim. $\square$

As a straightforward consequence we have

Corollary 3.1. *Let $(X_{\mathcal{L}}, \varphi, \Sigma)$ be a dynamical system. The set of all $\mathcal{A}_D$-extensions of $(X_{\mathcal{L}}, \varphi, \Sigma)$ is a lattice with the order $\leqslant_\Phi$.*

Now we shall study how two basic dynamical properties, transitivity and minimality, are carried over to $\mathcal{A}_D$-extensions. It is well know that these properties are carried over to factors and that, in general, these two concepts are in general different. For example, it is well known that the minimal sets in the unit interval $[0, 1]$ are Cantor sets or periodic orbits, but it is not difficult to construct transitive dynamical system on $[0, 1]$ (see, for example, [3]). At this point, it seems interesting to comment that in the study of how transitivity and minimality are carried over to an $\mathcal{A}_D$-extension $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ it is important that $K_{\mathcal{A}}$ have not isolated points. In fact, if the phase space has an isolated point, then transitivity implies that the system is just a single periodic orbit. In addition, if $x \in X$ and $k \notin \mathcal{O}_\varphi(x)$, then $k \notin \mathcal{O}_{\varphi_{\mathcal{A}}}(x)$, that is, when k is an isolated point of $K_{\mathcal{A}}$, the orbit of x by $\varphi_{\mathcal{A}}$ is not dense in $K_{\mathcal{A}}$. $\mathcal{A}$-extensions (in particular, $\mathcal{A}_D$-extensions) without isolated points were characterized in [9, Theorem 5] where it is proved that an $\mathcal{A}$-extension

$(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ of a dynamical system $(X_{\mathcal{L}}, \varphi, \Sigma)$ has no isolated points if and only if the C^*-algebra $\mathcal{A}$ contains no characteristic functions of singletons.

We turn our attention to transitivity and minimality of $\mathcal{A}_D$-extension. First we need the following lemmas.

Lemma 3.1. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$ and let $x \in X$. If $k \in i_{\mathcal{A}}^{-1}(x)$, then $\varphi_{\mathcal{A}}^n(k) \in i_{\mathcal{A}}^{-1}(\varphi^n(x))$ for every $n \in \Sigma$.*

Proof. The diagram

$$\begin{tikzcd} K_{\mathcal{A}} \arrow[r, "\varphi^n_{\mathcal{A}}"] \arrow[d, "i_{\mathcal{A}}"'] & K_{\mathcal{L}} \arrow[d, "i_{\mathcal{A}}"] \\ X_{\mathcal{L}} \arrow[r, "\varphi^n"] & X_{\mathcal{A}} \end{tikzcd}$$

commutes, because $i_{\mathcal{A}} \circ \varphi_{\mathcal{A}}^n|_X = \varphi^n \circ i$ and X is dense in $K_{\mathcal{A}}$. Then, for every $k \in i_{\mathcal{A}}^{-1}(x)$, we have

$$i_{\mathcal{A}}\big(\varphi_{\mathcal{A}}^n(k)\big) = i_{\mathcal{A}} \circ \varphi_{\mathcal{A}}^n(k) = \varphi^n \circ i_{\mathcal{A}}(k) = \varphi^n(x),$$

that is, $\varphi_{\mathcal{A}}^n(k) \in i_{\mathcal{A}}^{-1}(\varphi^n(x))$. $\square$

Lemma 3.2. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$. The following statements are equivalent*:

(1) *For every open set $\widetilde{V}$ in $K_{\mathcal{A}}$, there exists an open set U in $X_{\mathcal{L}}$ such that $U \subset \widetilde{V} \cap X$.*
(2) *$K_{\mathcal{A}}$ contains no isolated points.*

Proof. The implication (1) $\Rightarrow$ (2) is clear. To see (2) $\Rightarrow$ (1), let $\mathcal{A} = \langle C(X_{\mathcal{L}}) \cup B \rangle$ where $B \subset D(X_{\mathcal{L}})$ and let $\widetilde{V}$ be an open set in $K_{\mathcal{A}}$. Then there exist $f \in \mathcal{A}$ and an open set D in the complex plane such that $f^{-1}(D) \subset \widetilde{V}$, with $f^{-1}(D) = X \cap g^{-1}(D) \subset X \cap \widetilde{V} \subset \widetilde{V}$, where g is the extension of f to $K_{\mathcal{A}}$. We will consider two cases.

Case 1. $\emptyset \subsetneq f^{-1}(D) \subset D_f$. We shall check that this leads us to a contradiction. In fact, since D_f is a dispersed set, there exists $x \in f^{-1}(D)$ such that x is isolated in $f^{-1}(D)$. Then, there exists an open set U in $X_{\mathcal{L}}$ such that $U \cap f^{-1}(D) = \{x\}$. Consequently $U \cap g^{-1}(D) = \{x\}$, that is, x is isolated in $K_{\mathcal{A}}$, a contradiction.

Case 2. There exists $x \in f^{-1}(D)$ such that $x \notin D_f$. Since f is continuous in x, there exists an open set U in $X_{\mathcal{L}}$ such that $f(x) \in f(U) \subset D$. So $x \in U \subset f^{-1}(D) \subset \widetilde{V}$. $\square$

The foregoing lemmas permit us to relate the existence of isolated points in $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ with transitivity and minimality. The implication (2) $\Rightarrow$ (1) in the following theorem is a particular case of the well-known result that every topologically transitive dynamical system with an isolated point is a periodic orbit (so finite). We include the proof for the sake of completeness.

Theorem 3.2. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of a topologically transitive dynamical system $(X_{\mathcal{L}}, \varphi, \Sigma)$. The following statements are equivalent*:

(1) *$K_{\mathcal{A}}$ contains no isolated points.*
(2) *$(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ is topologically transitive.*

Proof. (1) $\Rightarrow$ (2) Let $\widetilde{U}, \widetilde{V}$ two nonempty open sets in $K_{\mathcal{A}}$. By Lemma 3.2 there exist two nonempty sets U and V in $X_{\mathcal{L}}$ with $U \subset \widetilde{U} \cap X$ and $V \subset \widetilde{V} \cap X$. Since $(X_{\mathcal{L}}, \varphi, \Sigma)$ is topologically transitive, there is a nonnegative integer n such that $\emptyset \subsetneq \varphi^n(U) \cap V \subset \varphi^n_{\mathcal{A}}(\widetilde{U}) \cap \widetilde{V}$. Thus, $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ is topologically transitive.

(2) $\Rightarrow$ (1) Suppose that there is an isolated point x in $K_{\mathcal{A}}$ and that $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ is topologically transitive. We shall prove that this leads us to a contradiction. To this in turn, notice that, given an open set $\widetilde{U}$ in $K_{\mathcal{A}}$, since $\{x\}$ is also open (in $K_{\mathcal{A}}$), the transitivity of $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ implies that there exist nonnegative integers m, n such that $x = \varphi^m_{\mathcal{A}}(x)$ and $\varphi^n_{\mathcal{A}}(x) \in \widetilde{U}$, that is, the orbit of x is a periodic orbit dense in $K_{\mathcal{A}}$. Thus, $K_{\mathcal{A}}$ is finite, a contradiction. $\square$

Theorem 3.3. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$. Consider the following statements*:

(1) *If there exists $x \in X$ such that $\mathcal{O}_\varphi(x)$ is dense in $(X_{\mathcal{L}}, \varphi, \Sigma)$, then $\mathcal{O}_{\varphi_{\mathcal{A}}}(k)$ is dense in $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ for every $k \in i^{-1}_{\mathcal{A}}(x)$.*
(2) *For every $x \in X$, x is not isolated in $K_{\mathcal{A}}$.*

Then (2) $\Rightarrow$ (1) *and, if the fiber of x by φ is finite,* (1) $\Rightarrow$ (2).

Proof. (2) $\Rightarrow$ (1) Let $\mathcal{A} = \langle C(X_{\mathcal{L}}) \cup B\rangle$ where $B \subset D(X_{\mathcal{L}})$. Let $\widetilde{V}$ be an open set in $K_{\mathcal{A}}$ and let $f \in \mathcal{A}$. By Lemma 3.2, there exists an open set U in $X_{\mathcal{L}}$ such that $U \subset f^{-1}(D_{r/2}(z_0)) \subset \widetilde{V}$ for some ball $D_r(z_0)$ of the complex plane.

Let $x \in X$ be such that $\mathcal{O}_\varphi(x)$ is dense, and let $k \in i^{-1}_{\mathcal{A}}(x)$. Then there exists $n \in \Sigma$ such that $\varphi^n(x) \in U$. Let $(x_\delta)_{\delta \in D}$ be a net in X converging to $\varphi^n_{\mathcal{A}}(k)$ in $K_{\mathcal{A}}$. By Lemma 3.1 the net $(x_\delta)_{\delta \in D}$ converges to $\varphi^n(x)$ in $X_{\mathcal{L}}$. Since U is an open set in $X_{\mathcal{L}}$, we can suppose without loss of generality, that the net $(x_\delta)_{\delta \in D}$ is contained in U. If g is the continuous extension of f to $K_{\mathcal{A}}$, a standard argument proves that

$$\bigl|g\bigl(\varphi^n_{\mathcal{A}}(k)\bigr) - z_0\bigr| < \frac{r}{2}.$$

That is, $\varphi^n_{\mathcal{A}}(k) \in g^{-1}(D_r(z_0))$. Thus, $\mathcal{O}_{\varphi_{\mathcal{A}}}(k)$ is dense in $K_{\mathcal{A}}$. Next, we shall assume that $\varphi^{-1}(x)$ is finite, for every $x \in X$ and we shall prove that (1) $\Rightarrow$ (2). For this in turn, suppose that x is an isolated point in $K_{\mathcal{A}}$ and let $x_1 \in X$ such that $\mathcal{O}_\varphi(x_1)$ is dense in $X_{\mathcal{L}}$. We shall conclude the proof by showing that there is a point with has no dense orbit. We shall consider several cases.

Case 1. $x = x_1$. First we shall show that $i^{-1}_{\mathcal{A}}(x)$ contains more than one point. In fact, if $i^{-1}_{\mathcal{A}}(x) = \{x\}$ then $i_{\mathcal{A}}(K_{\mathcal{A}} \setminus \{x\}) = X \setminus \{x\}$, that is x is isolated in $X_{\mathcal{L}}$, a contradiction. Now let $k \in i^{-1}_{\mathcal{A}}(x)$ with $k \neq x$. Since the orbit of $x_1 = x$ is dense, $\varphi^n(x) \neq x$ for every $n > 1$.

By Lemma 3.1 $\varphi^n_{\mathcal{A}}(k) \in i^{-1}_{\mathcal{A}}(\varphi^n(x))$, and consequently, $x \neq \varphi^n_{\mathcal{A}}(k)$ for every $n > 1$; that is, the orbit of k by $\varphi_{\mathcal{A}}$ is no dense.

Case 2. $x \notin \mathcal{O}_\varphi(x_1)$. This case is obvious.

Case 3. There exists $n > 1$ such that $x = \varphi^n(x_1)$. Since x is isolated in $K_\mathcal{A}$, the function $e_x \in \mathcal{A}$. Then, $e_x \circ \varphi^n \in \mathcal{A}$ is the characteristic function of $\varphi^{-n}(x)$. Let $h \in C(X_\mathcal{L})$ such that

$$h(x_1) = 1 \quad \text{and} \quad h\,|_{\varphi^{-n}(x)\setminus\{x_1\}} = 0.$$

Then, $h(e_x \circ \varphi^n) = e_{x_1}$, and we can reduce case 3 to case 1. $\square$

Corollary 3.2. *Let $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$. If φ is a homeomorphism, then* (1) *and* (2) *of Theorem* 3.3 *are equivalent.*

As a consequence of Theorems 3.2 and 3.3 we can obtain

Corollary 3.3. *Let $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$ without isolated points. Then*

(1) *If $(X_\mathcal{L}, \varphi, \Sigma)$ is topologically transitive, so is $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$.*
(2) *If $(X_\mathcal{L}, \varphi, \Sigma)$ is minimal, so is $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$.*

Corollary 3.4 [11, Corollary II.10]. *Let $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$ be an $\mathcal{A}$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$ with $\mathcal{A}$ a complex C^*-algebra which is topologically generated by a subset of functions each of which is continuous off a finite set, and has no characteristic functions of singletons. Then*

(1) *$(X_\mathcal{L}, \varphi, \Sigma)$ is topologically transitive, so is $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$.*
(2) *$(X_\mathcal{L}, \varphi, \Sigma)$ topologically minimal, so is $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$.*

4. Chaotic extensions

In this section we shall prove that if $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$ is an $\mathcal{A}_D$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$ without isolated points and $(X_\mathcal{L}, \varphi, \Sigma)$ is chaotic in Devaney's sense, so is $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$. We remind the reader that by the Devaney's definition of chaos (see [4]), a dynamical system (X, φ, Σ) is said to be *chaotic* if (1) it is topologically transitive, (2) the periodic points are dense in X, and (3) it has sensitive dependence on initial conditions.

Recall that a dynamical system $(X_\mathcal{L}, \varphi, \Sigma)$ is said to have sensitive dependence on initial conditions if there exists an entourage $U_\mathcal{L}$ in $\mathcal{L}$ such that, for every $x \in X_\mathcal{L}$ and every open set U in X with $x \in U$, there exist $x_U \in U$ and a positive integer n such that $(\varphi^n(x), \varphi^n(x_U)) \notin U_\mathcal{L}$.

It is known that if X is infinite (see [2,1]), then (1) and (2) together imply (3) in Devaney's definition. So, in order to study how the property of being chaotic in Devaney's sense is carried over $\mathcal{A}_D$-extensions, we need only to work with conditions (1) and (2).

Theorem 4.1. *Let $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_\mathcal{L}, \varphi, \Sigma)$ without isolated points. If $(X_\mathcal{L}, \varphi, \Sigma)$ has a dense subset of periodic points, so does $(K_\mathcal{A}, \varphi_\mathcal{A}, \Sigma)$.*

Proof. Let $C \subset X$ be a dense subset of periodics points for φ of $X_{\mathcal{L}}$ and let $\widetilde{V}$ be an open set in $K_{\mathcal{A}}$. By Lemma 3.2, there exists an open set U in $X_{\mathcal{L}}$ such that $U \subset X \cap \widetilde{V} \subset \widetilde{V}$. Since $U \cap C \neq \emptyset$ and $\varphi_{\mathcal{A}} |_X = \varphi$, C is also a dense subset of periodics points for $\varphi_{\mathcal{A}}$ of $K_{\mathcal{A}}$. □

Corollary 3.3 and Theorem 4.1 permits us to prove the promise result.

Corollary 4.1. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$ without isolated points. If $(X_{\mathcal{L}}, \varphi, \Sigma)$ is a chaotic dynamical system, so is $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$.*

We shall close the section turning our attention to the sensitive dependence on initial conditions. In general sensitive dependence on initial conditions is not transmitted by extensions (for an easy example see [2]), so your consideration has interest in itself. First we need the following lemma.

Lemma 4.1. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$ without isolated points. For every open set $\widetilde{V}$ in $K_{\mathcal{A}}$ and every $k \in \widetilde{V}$, there exist $k_{\widetilde{V}} \in \widetilde{V}$ with $k \neq k_{\widetilde{V}}$ and an open set U in $X_{\mathcal{L}}$ contained in $\widetilde{V}$ such that $i_{\mathcal{A}}(k_{\widetilde{V}}) \in U$.*

Proof. Let $\widetilde{V}$ be an open set in $K_{\mathcal{A}}$ and let $k \in \widetilde{V}$. Consider an open set $\widetilde{T}$ in $K_{\mathcal{A}}$ such that $k \in \widetilde{T} \subset \mathrm{cl}_{K_{\mathcal{A}}} \widetilde{T} \subset \widetilde{V}$. By Lemma 3.2, there exists an open set U in $X_{\mathcal{L}}$ such that $U \subset \widetilde{T} \cap X \subset \mathrm{cl}_{K_{\mathcal{A}}} \widetilde{T} \subset \widetilde{V}$. Choose an open set W in $X_{\mathcal{L}}$ such that $W \subset \mathrm{cl}_{X_{\mathcal{L}}} W \subset U \subset \mathrm{cl}_{K_{\mathcal{A}}} \widetilde{T} \subset \widetilde{V}$. Then $i_{\mathcal{A}}(\mathrm{cl}_{K_{\mathcal{A}}} W) \subset \mathrm{cl}_{X_{\mathcal{L}}}(i_{\mathcal{A}} W) = \mathrm{cl}_{X_{\mathcal{L}}} W \subset U$. The result follows from the fact that $\mathrm{cl}_{K_{\mathcal{A}}} W \setminus W$ is an infinite set. □

Theorem 4.2. *Let $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ be an $\mathcal{A}_D$-extension of $(X_{\mathcal{L}}, \varphi, \Sigma)$ without isolated points. If $(X_{\mathcal{L}}, \varphi, \Sigma)$ has sensitive dependence on initial conditions, so does $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$.*

Proof. Let $U_{\mathcal{L}}$ be an entourage in $\mathcal{L}$ for which $(X_{\mathcal{L}}, \varphi, \Sigma)$ has sensitive dependence on initial conditions. Since $i_{\mathcal{A}}$ is uniformly continuous, there exists an entourage $R_{\mathcal{A}}$ in $\widetilde{\mathcal{U}}_{\mathcal{A}}$ such that $R_{\mathcal{A}} \subset (i_{\mathcal{A}} \times i_{\mathcal{A}})^{-1}(U_{\mathcal{L}})$. Consider an entourage $W_{\mathcal{A}}$ such that $W_{\mathcal{A}} \circ W_{\mathcal{A}} \subset R_{\mathcal{A}}$. We shall prove that $W_{\mathcal{A}}$ is the entourage in $\widetilde{\mathcal{U}}_{\mathcal{A}}$ for which $(K_{\mathcal{A}}, \varphi_{\mathcal{A}}, \Sigma)$ has sensitive dependence on initial conditions.

Let $k \in K_{\mathcal{A}}$ and let $\widetilde{V}$ be an open set in $K_{\mathcal{A}}$ with $k \in \widetilde{V}$. By Lemma 4.1 there exists $k_{\widetilde{V}} \neq k$ in $\widetilde{V}$ and an open set U in $X_{\mathcal{L}}$ such that $i_{\mathcal{A}}(k_{\widetilde{V}}) = x_U \in U$. If there exists a positive integer n such that $(\varphi^n_{\mathcal{A}}(k), \varphi^n_{\mathcal{A}}(k_{\widetilde{V}})) \notin W_{\mathcal{A}}$ the proof is complete. Suppose that $(\varphi^n_{\mathcal{A}}(k), \varphi^n_{\mathcal{A}}(k_{\widetilde{V}})) \in W_{\mathcal{A}}$ for each positive integer n. Applying the definition of sensitive dependence on initial conditions to $U_{\mathcal{L}}$, $x_U \in X$ and U, there exist $x'_U \in U$ and a positive integer n such that $(\varphi^n(x_U), \varphi^n(x'_U)) \notin U_{\mathcal{L}}$. We claim that $(\varphi^n_{\mathcal{A}}(k), \varphi^n_{\mathcal{A}}(x'_U)) \notin W_{\mathcal{A}}$. Suppose contrary that $(\varphi^n_{\mathcal{A}}(k), \varphi^n_{\mathcal{A}}(x'_U)) \in W_{\mathcal{A}}$. Then, $(\varphi^n_{\mathcal{A}}(k_{\widetilde{V}}), \varphi^n_{\mathcal{A}}(x'_U)) \in W_{\mathcal{A}} \circ W_{\mathcal{A}} \subset R_{\mathcal{A}}$ and consequently

$$\begin{aligned}(i_{\mathcal{A}} \times i_{\mathcal{A}})\bigl((\varphi^n_{\mathcal{A}}(k_{\widetilde{V}}), \varphi^n_{\mathcal{A}}(x'_U))\bigr) &= \bigl(i_{\mathcal{A}}\bigl(\varphi^n_{\mathcal{A}}(k_{\widetilde{V}})\bigr), i_{\mathcal{A}}\bigl(\varphi^n_{\mathcal{A}}(x'_U)\bigr)\bigr)\\ &= \bigl(\varphi^n\bigl(i_{\mathcal{A}}(k_{\widetilde{V}})\bigr), \varphi^n_{\mathcal{A}}\bigl(x'_U\bigr)\bigr) = \bigl(\varphi^n(x_U), \varphi^n\bigl(x'_U\bigr)\bigr) \in U_{\mathcal{L}}\end{aligned}$$

which is a contradiction. Thus, $(\varphi^n_{\mathcal{A}}(k), \varphi^n_{\mathcal{A}}(x'_U)) \notin W_{\mathcal{A}}$ and the proof is complete. □

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